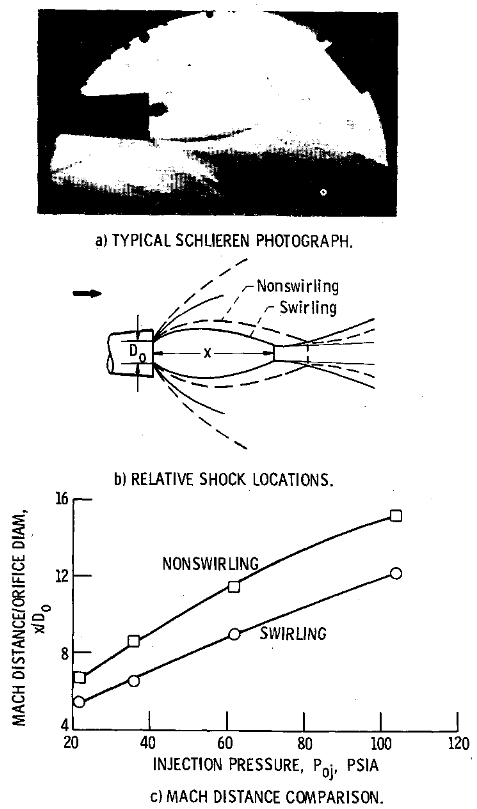

Fig. 4 Shock structure comparison.

$r/r_0 = 4$. This result indicates that the effect of swirl on the jet boundary position in supersonic flow is negligible.

Since the swirl was introduced 1 in. upstream of the jet exhaust some of the swirl may have decayed by the time the injectant reached the nozzle exhaust plane. Hence, an additional set of measurements was made using an insert located inside the injector, immediately upstream of the nozzle. This insert was a small plug having three spiral grooves on the periphery. These grooves made the injectant turn 135° over the length of the insert (0.64 cm). The measurements showed that the radial helium concentration profiles for the swirling injector with the insert were slightly lower ($\Delta r/r_0 = 0.25$) than the nonswirling injector profiles. Again this difference was equivalent to the inside diameter of the sampling probe. Hence, swirl does not give rise to accelerated mixing in supersonic flow.

In the course of this investigation, schlieren photographs were made of the base flow region of the injectors. A typical photo is shown in Fig. 4a. The highly underexpanded jet gives rise to the usual envelope (or barrel) structure which terminates in a Mach disc and two reflected waves. Slip lines can also be seen emanating from the reflection points. Figure 4b shows a comparison of the shock structure for both the swirling and nonswirling jets. It is noted that the distance to the Mach disk from the exit orifice is greater for the nonswirling jet. Figure 4c shows a comparison of this distance as a function of injection pressure. It has been shown that this distance is only a function of the static pressure ratio.[3] Hence, the curves shown in Fig. 4c indicate that the exit pressure of the swirling jet is less than that of the nonswirling jet. Since strong radial pressure and velocity gradients are an integral part of vortex flows it is conceivable that the jet exit pressure, integrated over the exit area, is less for the swirling jet than for the nonswirling one. The pressure decrease may be due to the increased viscous losses in the nozzle.

Since the jet boundary positions for both injectors were found to be identical, it must be concluded that the changes in shock structure are not indicative of a change in the 0% injectant boundary. It follows that the change in shock position indicates a change in the streamlines within the flow. The radial distribution of the injectant should therefore be different for the swirling and nonswirling cases. The experimental data, shown in Fig. 2, show that the radial distributions are, in fact, dissimilar.

The Mach disk position has an important implication as far as injection normal (rather than coaxial) to the stream is concerned. This is so because the distance from the injection orifice to the Mach disk position is proportional to the amount of jet penetration obtained in supersonic streams. Hence it appears that swirling jet flow would yield less penetration than nonswirling flow when injection normal to the freestream is employed.

In conclusion, the concept of accelerated mixing in supersonic streams due to swirl has been tested. The results indicate that swirl does not produce any enhancement of mixing.

Note Added in Proof

Since this Note was prepared for publication, a report[4] has been published dealing with the effect of swirl on co-axial jet mixing in a supersonic stream. From a comparison of nonswirl and swirl data, it was concluded in Ref. 4 that the swirl had no discernible effect on the mixing. That conclusion is in agreement with the results of this study.

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Best Finite Elements Distribution around a Singularity

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Introduction

As long as the solution, both in boundary value problems and eigenvalue problems, does not include singularities, increasing the degree of the interpolation functions inside the elements will always result in an increase in the rate of convergence of the finite element method. In fact,[1] if the interpolation (shape) functions inside the element include a complete set of polynomials of degree p then for problems of the $2m$th order ($m = 1$ in harmonic problems and $m = 2$ in biharmonic problems) the error in the energy in boundary value problems and the error in the eigenvalues in eigenvalue problems is $0[h^{2(p+1-m)}]$ where h is the diameter of the element. The reason for this is that any smooth function (without singularities) can be approximated as closely as desired by polynomials. If, however, the solution function includes a singularity then around the singular point the function itself or its derivatives up from a certain degree can no more be approximated by polynomials. Increasing the order of the interpolation functions inside the element will not result, with a given mesh, in

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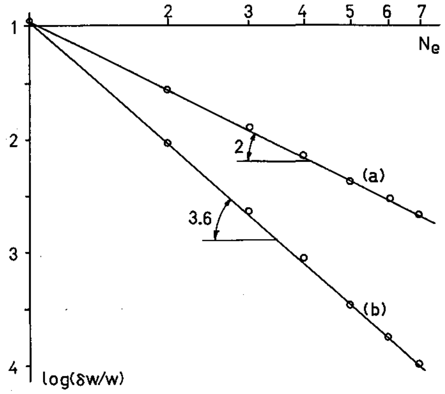

Fig. 1 Rate of convergence of the central deflection w of a clamped circular plate point loaded at the center, for a) a uniform mesh and b) a quadratically spaced mesh.

an indefinite increase in the rate of convergence of the method.

An obvious way to overcome this limitation is to include [2,3] in the interpolation functions a function of the form of the singularity. This encumbers, however, the finite element method. But with polynomial elements the most severe loss of accuracy occurs near the singularity and the question arises then whether or not the loss of accuracy near the singularity can be recompensated by reducing the size of the elements near the singularity so as to regain the full rate of convergence of the method. It is the purpose of this note to demonstrate that indeed by a proper spacing of the mesh near the singularity the full rate of convergence can be obtained with polynomial elements.

Boundary Value Problem

Consider a boundary value problem of the 2mth order with a solution including a singularity of the form $u = r^\alpha$ where α is a fraction. Since the singularity is due only to r attention is restricted to the n dimensional unit sphere and the energy norm

$$\|u\|_m^2 = \int_o^1 \left(\frac{d^m u}{dr^m}\right)^2 r^{n-1}\, dr \tag{1}$$

Consideration is also limited here to such α for which $\|u\|_m < \infty$, or for α such that

$$\alpha > m - n/2 \tag{2}$$

If the solution does not include singularities then from Taylor's theorem it results that the finite element interpolate $\tilde{u}$ (the function $\tilde{u}$ that agrees at the nodal points with the true solution u) can approximate u such that the error in the energy is $0[h^{2(p+1+m)}]$ Since the finite element (Rayleigh-Ritz-Galerkin) method chooses the solution that minimizes the error in the energy, the error in the interpolate constitutes an upper bound on the error in the finite element solution. In a more precise manner, if $\tilde{u}$ is the interpolate to u and $\hat{u}$ is the finite element solution, then

$$\|u - \tilde{u}\|_m \geqq \|u - \hat{u}\|_m \tag{3}$$

The problem of estimating the rate of convergence of the finite element method is reduced thereby to estimating the error in the finite element interpolate.

For functions of the form $u = r^\alpha$ which are in fact one dimensional a finite element interpolate can readily be constructed and the rate of convergence of the method estimated from a direct calculation of the error in the interpolate. Another possibility is to use the approximation formula[4]

$$\|u - \tilde{u}\|_m^2 \leqq ch^{2(p+1-m)}\|u\|_{p+1} \tag{4}$$

where c is independent of h.

Assume now that $u = r^\alpha$ and that the interval $0 \leqq r \leqq 1$ is divided equally into N_{er} finite elements such that the i-th element is situated now between $r = (i-1)h$ and $r = ih$. The error in the i-th element is given by

$$\|u - \tilde{u}\|_m^2 \leqq ch^{2(\alpha-m)+n}\, i^{2(\alpha-p)-3+n} \tag{5}$$

where in Eq.(5) it was assumed that $i > 1$. It results from Eqs.(5) and (3) that the error in finite element solution $\hat{u}$ is such that

$$\|u - \hat{u}\|_m^2 \leqq ch^{2(\alpha-m)+n} \sum_{i=1}^{\infty} i^{2(\alpha-p)-3+n} \tag{6}$$

and the sum in Eq.(6) converges for $2(\alpha - p) - 3 + n < -1$. Therefore increasing p beyond

$$p = \alpha - 1 + n/2 \tag{7}$$

will not increase the rate of convergence

$$\|u - \hat{u}\|_m^2 \leqq ch^{2(\alpha-m)+n} \tag{8}$$

Another type of singularity is of the form $r^\alpha \log(r)$. But as $\log(r)$ grows slower than r^ε for any $\varepsilon > 0$, $r^\alpha \log(r)$ behaves essentially like r^α.

The prediction of Eq.(8) is also confirmed numerically on a plate bay a point force. In this case, the singularity is of the form $r^2 \log(r)$ and hence, according to Eq. (8), since $\alpha = 2$, $m = 2$, and $n = 2$, a uniform mesh will not produce a rate of convergence larger than $0(h^2)$ which was indeed confirmed numerically[5]. With no singularities the quartic ($p = 4$) plate bending element used in these experiments could have yielded an energy error $0(h^6)$ which was also confirmed by these experiments.

A uniform mesh is however largely inefficient since the error is concentrated near the singularity. It will be shown now that by reducing the mesh size near the singularity the full rate of energy convergence $0[h^{2(p+1-m)}]$ can be regained.

Assume for this, that the interval $0 \leqq r \leqq 1$ is divided into N_{er} finite elements of diameter $h, \alpha_1 h, \ldots, \alpha_{N_{er}} h$ such that

$$h(1 + \alpha_1 + \ldots + \alpha_{Ner}) = 1 \tag{9}$$

Eq. (5) predicts that an equal energy error in each element is obtained with

$$\alpha_i^{2(\alpha-m)+n}\, i^{2(\alpha-p)-3+n} = 1 \tag{10}$$

or

$$\alpha = i^z, \quad z = [2(p - \alpha) + 3 - n]/[2(\alpha - m) + n] \tag{11}$$

Summing the errors over-all the N_{er} finite elements, the global error in the energy becomes

$$\|u - \hat{u}\|_m^2 \leqq c\, N_{er}\, h^{2(\alpha-m)+n} \tag{12}$$

where h is now the diameter of the smallest element near the singular point. Since α_i grows like i^z the sum of α_i over N_{er} finite elements grows like N_{er}^{z+1} and Eq. (9–12) yield finally

$$\|u - \hat{u}\|_m^2 \leqq c\, N_{er}^{-2(p+1-m)} \tag{13}$$

and the full rate of convergence is regained.

Eigenproblems

The situation in eigenproblems is entirely anologous to that in boundary value problems. For eigenproblems of the $2m$-th order, inclusion in the interpolation functions a complete set of polynomials of degree p, assures[6] that the error $\delta\lambda_r$ in the r-th eigenvalue λ_r is such that

$$\delta\lambda_r/\lambda_r \leqq c\, N_{er}^{-2(p+1-m)}(\lambda_r/\lambda_1)^{(p+1-m)/m} \tag{14}$$

If the eigenfunctions include a singularity of the form r^α [or $r^\alpha \log(r)$] the rate of convergence (14) can be obtained by spacing the mesh of finite elements according to Eq. (11).

Numerical Examples

The prediction of Eq. (11) was tested numerically on a clamped circular plate, point loaded at the center, and discretized by cubic ($p = 3$) elements. The singularity in this case is of the form $r^2 \log(r)$ and therefore with $\alpha = 2$, $m = 2$, and $n = 2$, Eq. (8) predicts that for a uniform mesh the highest rate of convergence is $0(h^2)$. For obtaining the full rate of convergence $0(h^4)$ obtainable with cubic elements, Eq. (11) predicts that the mesh should by varied quadratically ($z = 2$). As seen from Fig. 1 this maximal rate of convergence is almost obtained. Also of interest is the

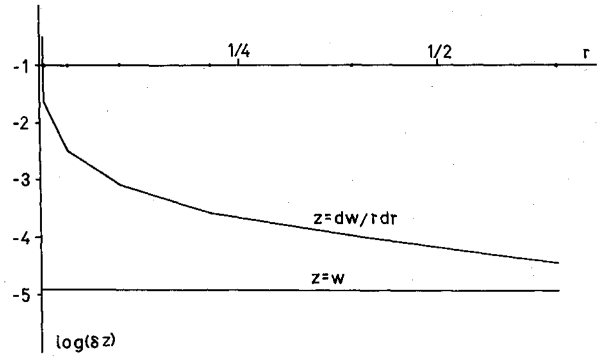

Fig. 2 Clamped circular plate point loaded at the center and discretized with quadratically varying seven elements. Figure shows distribution of errors in w and dw/rdr along the radius r.

pointwise error both in the displacements and strains. Figure 2 depicts the error δw in the displacements w, and the error $\delta(dw/rdr)$ in the circumferential strain, along the radius r of the plate. Note that the error in the strain is confined to the element nearest to the singularity.

In the previous example the exact solution was of the form $r^2(1 + \log r)$. With the cubic ($p = 3$) elements used there is no errors in approximating r^2, the only errors being due to the approximation of $r^2 \log r$. In the more general case, the finite element trial function must approximate both the regular and singular components of the solution. In such a case, the mesh distribution is composed of a uniform (or any better) mesh for approximating the regular portion of the solution plus an exponentially varying mesh for the singular portion.

A problem of this type is provided by a vibrating plate with a point mass at the center.[7] The exact nature of the singularity is not readily available, but since this problem is so closely related to the previous one, also here the singularity was assumed to require a quadratically varying mesh. This mesh distribution was superposed on a uniform mesh and the convergence of the first eigenvalue λ_1 for a clamped plate with a central point mass 0.1 of the total plate mass, is shown in Fig. 3.

Conclusions

Consider a boundary value problem or an eigenvalue problem of the 2mth order with a solution including a singularity of the form r^α[or $r^\alpha \log(r)$]. Let this problem be defined in n dimensions and be discretized by finite elements inside which the interpolation (shape) functions include a complete polynomial of degree p. The full rate of convergence can be obtained with polynomial elements if they are spaced along r such that the diameter of the i-th element is given by

$$h_i = hi^z, \quad z = [2(p-\alpha)+3-n]/[2(\alpha-m)+n] \qquad (15)$$

It should not escape ones attention that large mesh ratios produce ill-conditioned matrices.[8]

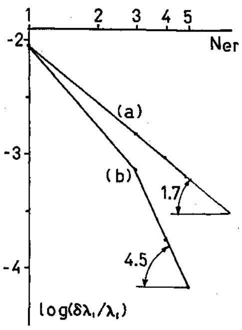

Fig. 3 Clamped circular plate with point mass at the center, ratio of central mass to plate mass being 0.1. Figure shows convergence of the first eigenvalue λ_1 for a) a uniform mesh and b) a quadratically varying mesh superposed on a uniform mesh.

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Allowable Regions for Stability Multiplier Characteristics

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Introduction

FOLLOWING Popov's[1] innovatory frequency-domain criterion for the absolute stability of systems containing a linear part transfer function $G(s)$ and a nonlinearity in cascade in a negative feedback loop, a number of interesting stability criteria[2–4]‡ have appeared in the control theory literature. All these criteria aim at relaxing the conditions on $G(s)$ corresponding with the imposition of restrictions such as monotonicity, odd property etc. on the nonlinear function and take the following form. For the absolute stability of the system containing a nonlinearity belonging to a certain class $\mathscr{F}_c$, it is sufficient if there exists a function $Z(s)$ belonging to an associated class $\mathscr{Z}_c$ and satisfying the two necessary conditions,

$$Re\, Z(j\omega) \geqq 0 \quad \forall\, \omega \in [0, \infty), \quad Z(j\omega) \not\equiv 0 \qquad (1)$$

$$Re\, Z(j\omega)\, G(j\omega) \geqq 0 \quad \forall\, \omega \in [0, \infty) \qquad (2)$$

A function $Z(s)$ which satisfies Eqs. (1) and (2) is called a "stability multiplier" for the system. However, the applicability of these criteria is largely restricted by the fact that these do neither give a procedure for determining the multiplier $Z(s)$ nor indicate the possibility of the existence of such a function for a given problem. This Note is aimed at removing some of the difficulties involved in the determination of stability multipliers for a given $G(s)$.

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‡ These are only a few of the prominent results.